Access to Science, Engineering and Agriculture: Mathematics 1 MATH00030 Chapter 5 Solutions

1. (a) We will first use the fact that the sine function repeats every 2π.
   Thus $\sin\left(\dfrac{5\pi}{3}\right) = \sin\left(\dfrac{5\pi}{3} - 2\pi\right) = \sin\left(-\frac{\pi}{3}\right)$.
   We can now use our table of common values and $\sin(-\theta) = -\sin(\theta)$ to obtain $\sin\left(-\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$. Hence $\sin\left(\dfrac{5\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$.

   (b) We will first use the fact that the cosine function repeats every 2π.
   Thus $\cos\left(\dfrac{11\pi}{6}\right) = \cos\left(\dfrac{11\pi}{6} - 2\pi\right) = \cos\left(-\frac{\pi}{6}\right)$.
   We can now use our table of common values and $\cos(-\theta) = \cos(\theta)$ to obtain $\cos\left(-\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$. Hence $\cos\left(\dfrac{11\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$.

   (c) Here we will first use the fact that the tangent function repeats every π.
   Thus $\tan\left(\dfrac{5\pi}{6}\right) = \tan\left(\dfrac{5\pi}{6} - \pi\right) = \tan\left(-\frac{\pi}{6}\right)$.
   We can now use our table of common values and $\tan(-\theta) = -\tan(\theta)$ to obtain $\tan\left(-\frac{\pi}{6}\right) = -\tan\left(\frac{\pi}{6}\right) = -\dfrac{1}{\sqrt{3}}$. Hence $\tan\left(\dfrac{5\pi}{6}\right) = -\dfrac{1}{\sqrt{3}}$.

2. (a) Here we will first use $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$.
   We have $\sin\left(\dfrac{3\pi}{4}\right) = \cos\left(\dfrac{\pi}{2} - \dfrac{3\pi}{4}\right) = \cos\left(-\frac{\pi}{4}\right)$.
   Next we will use $\cos(-\theta) = \cos(\theta)$ and our table of common values to obtain $\cos\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$. Hence $\sin\left(\dfrac{3\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$.

   (b) Here we will first use $\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$.
   We have $\cos\left(\dfrac{2\pi}{3}\right) = \sin\left(\dfrac{\pi}{2} - \dfrac{2\pi}{3}\right) = \sin\left(-\frac{\pi}{6}\right)$.
   Next we will use $\sin(-\theta) = -\sin(\theta)$ and our table of common values to obtain $\sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\dfrac{1}{2}$. Hence $\cos\left(\dfrac{2\pi}{3}\right) = -\dfrac{1}{2}$.

   (c) Here we will first use $\tan(\theta) = \cot\left(\frac{\pi}{2} - \theta\right)$.
   We have $\tan\left(\dfrac{5\pi}{6}\right) = \cot\left(\dfrac{\pi}{2} - \dfrac{5\pi}{6}\right) = \cot\left(-\frac{\pi}{3}\right)$.
   Next we will use $\cot(-\theta) = -\cot(\theta)$, the definition of cotangent and our table

of common values to obtain $\cot\left(-\frac{\pi}{3}\right) = -\cot\left(\frac{\pi}{3}\right) = -\frac{1}{\tan\left(\frac{\pi}{3}\right)} = -\frac{1}{\sqrt{3}}$.

Hence $\tan\left(\frac{5\pi}{6}\right) = -\frac{1}{\sqrt{3}}$.

3. (a) If we solve $a^2 = b^2+c^2-2bc\cos(A)$ for $\cos(A)$ we obtain $\cos(A) = \frac{b^2+c^2-a^2}{2bc}$ and on substituting for a, b and c, $\cos(A) = \frac{5^2+4^2-6^2}{2(5)(4)} = \frac{1}{8}$.
Hence $A \simeq 82.82°$.
Next, on solving $b^2 = a^2+c^2-2ac\cos(B)$ for $\cos(B)$, we obtain $\cos(B) = \frac{a^2+c^2-b^2}{2ac}$ and on substituting for a, b and c,
$\cos(B) = \frac{6^2+4^2-5^2}{2(6)(4)} = \frac{9}{16}$. Hence $B \simeq 55.77°$.
Now, using the fact that the angles in a triangle sum to 180°, we have $C \simeq 180° - 82.82° - 55.77° \simeq 41.41°$.

(b) If we solve $a^2 = b^2+c^2-2bc\cos(A)$ for $\cos(A)$ we obtain $\cos(A) = \frac{b^2+c^2-a^2}{2bc}$ and on substituting for a, b and c, $\cos(A) = \frac{5^2+3^2-5^2}{2(5)(3)} = \frac{3}{10}$.
Hence $A \simeq 72.54°$.
Next, on solving $b^2 = a^2+c^2-2ac\cos(B)$ for $\cos(B)$, we obtain $\cos(B) = \frac{a^2+c^2-b^2}{2ac}$ and on substituting for a, b and c,
$\cos(B) = \frac{5^2+3^2-5^2}{2(5)(3)} = \frac{3}{10}$. Hence we also have $B \simeq 72.54°$.
Note that this also follows since the triangle is an isosceles triangle (it has two equal sides and hence two equal angles).
Now, using the fact that the angles in a triangle sum to 180°, we have $C \simeq 180° - 72.54° - 72.54° \simeq 34.92°$.

(c) Using the sine rule in the form $\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$, we obtain
$\frac{\sin(A)}{8} = \frac{\sin(44°)}{10}$. Thus $\sin(A) = \frac{8\sin(44°)}{10} \simeq 0.5557$.
Hence $A \simeq 33.76°$ or $A \simeq 180° - 33.76° = 146.24°$. We now have to decide which of these values is correct. Suppose that $A \simeq 146.24°$. Then the two angles we know add up to approximately $44° + 146.24° = 190.24°$ and this is too many degrees for a triangle. Hence we now know that $A \simeq 33.76°$.
Next, using the fact that the angles in a triangle sum to 180°, we have $C \simeq 180° - 44° - 33.76° \simeq 102.24°$.
Finally we will find c using the cosine rule in the form $c^2 = a^2+b^2-2ab\cos(C)$.
We have $c^2 \simeq 8^2 + 10^2 - 2(8)(10)\cos(102.24°) \simeq 197.92$.
Thus $c \simeq \sqrt{197.92} \simeq 14.07$.
Note that in questions like this, where there are a series of calculations which depend on earlier ones, it is important to use full calculator accuracy in the intermediate calculations (even though the answers are only written down to a lower accuracy) since otherwise large rounding errors can occur.

(d) Using the sine rule in the form $\dfrac{\sin(A)}{A} = \dfrac{\sin(B)}{b}$, we obtain $\dfrac{\sin(A)}{7} = \dfrac{\sin(38°)}{12}$. Thus $\sin(A) = \dfrac{7\sin(38°)}{12} \simeq 0.3591$.
Hence $A \simeq 21.05°$ or $A \simeq 180° - 21.05° = 158.95°$. We now have to decide which of these values is correct. Suppose that $A \simeq 158.95°$. Then the two angles we know add up to approximately $38° + 158.95° = 196.95°$ and this is too many degrees for a triangle. Hence we now know that $A \simeq 21.05°$.

Next, using the fact that the angles in a triangle sum to 180°, we have $C \simeq 180° - 38° - 21.05° \simeq 120.95°$.

This time (to show a different method) we will find c using the sine rule in the form $\dfrac{c}{\sin(C)} = \dfrac{b}{\sin(B)}$. Hence $c = \dfrac{b\sin(C)}{\sin(B)} \simeq \dfrac{12\sin(120.95°)}{\sin(38°)} \simeq 16.72$.

(e) In this case we can use the cosine rule in the form $a^2 = b^2 + c^2 - 2bc\cos(A)$ to obtain $a^2 = 9^2 + 12^2 - 2(9)(12)\cos(133°) \simeq 372.31$. Hence $a \simeq \sqrt{372.31} \simeq 19.30$.

Next, on solving $b^2 = a^2 + c^2 - 2ac\cos(B)$ for $\cos(B)$, we obtain
$\cos(B) = \dfrac{a^2 + c^2 - b^2}{2ac}$ and on substituting for a, b and c,
$\cos(B) \simeq \dfrac{19.30^2 + 12^2 - 9^2}{2(19.30)(12)} \simeq 0.9400$. Hence $B \simeq 19.95°$.
Finally, using the fact that the angles in a triangle sum to 180°, we have $C \simeq 180° - 133° - 19.95° \simeq 27.05°$.

(f) In this case we can use the cosine in the form $a^2 = b^2 + c^2 - 2bc\cos(A)$ to obtain $a^2 = 8^2 + 9^2 - 2(8)(9)\cos(70°) \simeq 95.75$. Hence $a \simeq \sqrt{95.75} \simeq 9.79$.

Next, on solving $b^2 = a^2 + c^2 - 2ac\cos(B)$ for $\cos(B)$, we obtain
$\cos(B) = \dfrac{a^2 + c^2 - b^2}{2ac}$ and on substituting for a, b and c,
$\cos(B) \simeq \dfrac{9.79^2 + 9^2 - 8^2}{2(9.79)(9)} \simeq 0.6401$. Hence $B \simeq 50.20°$.
Finally, using the fact that the angles in a triangle sum to 180°, we have $C \simeq 180° - 70° - 50.20° \simeq 59.80°$.

4. (a) Here we will use $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$ with $A = \dfrac{\pi}{4}$ and $B = \dfrac{\pi}{6}$. Hence

$$\begin{aligned}
\sin\left(\frac{5\pi}{12}\right) &= \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) \\
&= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) \\
&= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{3}+1}{2\sqrt{2}}.
\end{aligned}$$

(b) Here we will use $\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$ with $A = \frac{\pi}{6}$ and $B = \frac{\pi}{4}$. Hence

$$\begin{aligned}\sin\left(-\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\\ &= \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)\\ &= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)\\ &= \frac{1-\sqrt{3}}{2\sqrt{2}}.\end{aligned}$$

(c) Here we will use $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ with $A = \frac{\pi}{4}$ and $B = \frac{\pi}{3}$.

$$\begin{aligned}\cos\left(\frac{7\pi}{12}\right) &= \cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right)\\ &= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)\\ &= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{1-\sqrt{3}}{2\sqrt{2}}.\end{aligned}$$

(d) Here we will use $\tan(A-B) = \dfrac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$ with $A = \frac{\pi}{3}$ and $B = \frac{\pi}{4}$.

$$\tan\left(\frac{\pi}{12}\right) = \frac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{4}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{4}\right)} = \frac{\sqrt{3}-1}{1 + \left(\sqrt{3}\right)(1)} = \frac{\sqrt{3}-1}{\sqrt{3}+1}.$$

We can also simplify this as follows:

$$\frac{\sqrt{3}-1}{\sqrt{3}+1} \cdot \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{3 - 2\sqrt{3} + 1}{3-1} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}.$$

I will give full marks for $\tan\left(\frac{\pi}{12}\right) = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ however.

5. (a) Using $\sin^2(\theta) = \dfrac{1 - \cos(2\theta)}{2}$ with $\theta = \frac{\pi}{12}$, we have

$$\sin^2\left(\frac{\pi}{12}\right) = \frac{1 - \cos\left(2 \times \frac{\pi}{12}\right)}{2} = \frac{1 - \cos\left(\frac{\pi}{6}\right)}{2} = \frac{1 - \frac{\sqrt{3}}{2}}{2} = \frac{2-\sqrt{3}}{4}.$$

Hence $\sin\left(\frac{\pi}{12}\right) = \sqrt{\dfrac{2-\sqrt{3}}{4}}$.

(b) Using $\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$ with $\theta = \dfrac{\pi}{8}$, we have

$$\cos^2\left(\frac{\pi}{8}\right) = \frac{1+\cos\left(2\times\frac{\pi}{8}\right)}{2} = \frac{1+\cos\left(\frac{\pi}{4}\right)}{2} = \frac{1+\frac{1}{\sqrt{2}}}{2} = \frac{\sqrt{2}+1}{2\sqrt{2}}.$$

Hence $\cos\left(\dfrac{\pi}{8}\right) = \sqrt{\dfrac{\sqrt{2}+1}{2\sqrt{2}}}$.

(c) Using $\tan^2(\theta) = \dfrac{1-\cos(2\theta)}{1+\cos(2\theta)}$ with $\theta = \dfrac{\pi}{8}$, we have

$$\tan^2\left(\frac{\pi}{8}\right) = \frac{1-\cos\left(2\times\frac{\pi}{8}\right)}{1+\cos\left(2\times\frac{\pi}{8}\right)} = \frac{1-\cos\left(\frac{\pi}{4}\right)}{1+\cos\left(\frac{\pi}{4}\right)} = \frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}-1}{\sqrt{2}+1}.$$

Hence $\tan\left(\dfrac{\pi}{8}\right) = \sqrt{\dfrac{\sqrt{2}-1}{\sqrt{2}+1}}$.

We can also simplify this as follows:

$$\frac{\sqrt{2}-1}{\sqrt{2}+1} = \frac{\sqrt{2}-1}{\sqrt{2}+1}\cdot\frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{2-2\sqrt{2}+1}{2-1} = 3-2\sqrt{2},$$

so that $\tan\left(\dfrac{\pi}{8}\right) = \sqrt{3-2\sqrt{2}}$.

In fact this can be further simplified to $\tan\left(\dfrac{\pi}{8}\right) = \sqrt{2}-1$

(it is easy to see $(\sqrt{2}-1)^2 = 3-2\sqrt{2}$ but not the other way around).

I will give full marks for $\tan\left(\dfrac{\pi}{8}\right) = \sqrt{\dfrac{\sqrt{2}-1}{\sqrt{2}+1}}$ however.